

Simplicial James-Hopf map and decompositions of the unstable Adams spectral sequence for suspensions

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We use combinatorial group theory methods to extend the definition of a classical James-Hopf invariant to a simplicial group setting. This allow us to realize certain coalgebra idempotents at \mathbf{sSet}_* -level and obtain a functorial decomposition of the spectral sequence, associated with the lower p -central series filtration on the free simplicial group.

[55P35](#); [55T15](#), [18G30](#)

1 Introduction

One of the ultimate goals in unstable homotopy is to understand how the loops-over-suspension functor $\Omega\Sigma$ changes the homotopy type of the space. One aspect of such understanding is the computation and investigation of the homotopy groups of loop spaces and one of the most powerful methods for it is an unstable Adams spectral sequence first introduced in [1]. Construction of this spectral sequence starts with a free simplicial group model for a loop space, equipped with a lower p -central series filtration, which produces the spectral sequence itself. So framework for this machinery is a category of (pointed) simplicial sets \mathbf{sSet}_* . Computations show that for a loop space over spheres, p -locally unstable Adams spectral sequence is accelerated in a sense that non-trivial elements on its first page are concentrated in columns with numbers p^k (see [1]).

Another approach to investigating loop-suspension functor $\mathrm{Top}_* \rightarrow \mathrm{Top}_*$ through functorial direct product decompositions was introduced in [14]. It uses significantly a certain model for $\Omega\Sigma X$ in Top_* - James construction JX - a reduced free monoid on points of X together with a word-length filtration $J_k X$ on it. Among all (p -local) decompositions of the form

$$(1) \quad \Omega\Sigma X \simeq_p A \times B$$

there is a minimal one, called A^{\min} in a sense that it is obtained from a minimal coalgebra retract of a tensor algebra functor T - which is homological representation

of an $\Omega\Sigma$. It can be shown that primitive elements of such minimal coalgebra retract are concentrated in the degrees p^k .

The aim of the present paper is to translate the above mentioned decompositions to a category of simplicial sets and furthermore to extend them to a level of spectral sequences. From this point of view primitive elements of A^{min} form a first page of a functorial sub-spectral sequence of unstable Adams spectral sequence for suspensions and therefore result about acceleration of spectral sequence for ΩS^{n+1} becomes a particular case (since $A^{min}(S^n) = S^n$) of a more general situation.

We will briefly recall the idea of the loop-suspensions decompositions as in [14]. Let $\mathbf{k} = \mathbb{Z}/p$ be a ground field, then by Bott-Samelson theorem homology of $\Omega\Sigma X$ with coefficients in k is given by the tensor algebra $T(\bar{H}_*(X))$ on the reduced homology on X , i.e. functorially there is a map

$$(2) \quad [\Omega\Sigma, \Omega\Sigma] \xrightarrow{H_*} \text{Hom}_{\text{Coalg}}(T, T)$$

By geometric realization theorem in [14] this map is surjective, and in fact all natural coalgebra maps $f : T \rightarrow T$ can be realized as an elements in a certain subgroup of $[J, \Omega\Sigma]$ called Cohen group H_∞ first introduced in [2]. Now given any coalgebra idempotent $f : T \rightarrow T$ it can be lifted to a space level to obtain a map $\varphi : \Omega\Sigma \rightarrow \Omega\Sigma$, $\varphi_* = f$ and $A = \text{hocolim } \varphi$ will give a first piece of decomposition (1), second piece is given similarly by a homotopy mapping telescope of a complement map $[\text{id}]\varphi^{-1}$.

To adapt this picture to sSet_* , one need a suitable analogue of the Cohen group H_∞ which will realize a coalgebra idempotents on the level of simplicial sets. In Top_* Cohen group is defined as a subgroup of $[J, \Omega\Sigma]$, generated as a pro-group by compositions

$$(3) \quad JX \xrightarrow{H_k} JX^{\wedge k} \xrightarrow{J(\bar{\Delta})} JX^{\wedge l} \xrightarrow{J(\sigma)} J(X^{\wedge l}) \xrightarrow{W_l} \Omega\Sigma X$$

first and a last map here is combinatorial James-Hopf map and Whitehead product respectively (see definition (2.1) below). The most straightforward way to translate H_∞ to a sSet_* is to pass all objects through geometric realization/singular chains functors. Unfortunately, heaviness construction of $|-| - \text{Sing}$ adjunction hides combinatorial nature of H_∞ , so preservation of lower p -central series by its elements looks untraceable under this approach.

Instead we will define generators of Cohen group as a certain natural transformations of functors on pointed sets and will extend this definition to a sSet_* level-wisely. The main object of interest here is Milnor's construction $F[-]$ - simplicial group model for a

loop-suspension functor $\Omega\Sigma$. Our first goal is to extend the combinatorial James-Hopf map from free monoid to a free group framework. It is achieved by translating Hilton-Hopf map to simplicial level and applying Hall's commutator collection process inside simplicial version of Hilton-Milnor's theorem. The resulting definition (2.2) have all desired properties of a classical James-Hopf map and moreover can be considered as a non-abelian version of Magnus embedding [9]. Since everything happens level-wisely, Whitehead product can be considered just as iterated commutator map, permutations and iterated reduced diagonal maps stays the same as in (3).

With help of Fox differential calculus it will be shown that elements of Cohen group preserve lower central series on free simplicial group $F[-]$ and therefore induce a map between lower $(p-)$ central towers of fibrations. Extending these maps to an exact couple level will equip unstable Adams spectral sequence with an action of Cohen group.

Such action used to obtain a functorial decomposition of this spectral sequence in a following way. As before, one starts with a functorial coalgebra idempotent $f : T \rightarrow T$, which is now extended to a natural self-transformation of functor $T(\mathbf{k}[-]) : \mathbf{sSet}_* \rightarrow \mathbf{sCoalg}_{\mathbf{k}}$. By simplicial realization theorem (4.3) this natural transformation can be lifted to a self-map of l.c.s. tower and each level of the corresponding tower can be decomposed through a mapping telescope construction in a usual way. Such decomposition preserves the fibrations inside the l.c.s. tower, which can be therefore presented as a direct product of two towers, with layers weak equivalent to primitive elements of coalgebra retracts of $T(\mathbf{k}[-])$, corresponded to idempotent f and its complement (theorem (4.6)). Choosing an idempotent which corresponds to a minimal coalgebra retract (A^{min} subfunctor of T) will produce a direct summand of l.c.s. tower with non-trivial layers only at degrees p^k . Therefore corresponding spectral sequence will be accelerated in the same sense as in [1].

The paper is organized as follows. Extension of the James-Hopf invariants to the simplicial group level (definition (2.2)) together with motivation of such extension and connection with Magnus embedding (theorem (2.4)) is given in Section 2. Preservation of lower p -central series by elements of Cohen group discussed in theorem (3.8) and the construction itself is given in definition (22) of Section 3. Section 4 is devoted to applications of action of Cohen group on spectral sequence (3.10): decomposition of spectral sequence (4.6) and (4.7)) and acceleration of the part that correspond to a minimal coalgebra decomposition (4.8)

2 James-Hopf map for Milnor's construction

Set_* will denote the category of pointed sets (with distinct point $*$). In this category *wedge sum* $X \vee Y = X \cup Y / *_X \sim *_Y \hookrightarrow X \times Y$ and *smash product* $X \wedge Y = \frac{X \times Y}{X \vee Y}$ of pointed spaces X and Y are defined. Category of groups Grp will also be considered pointed, with identity element of each object as a distinct point. Then there is a pair of adjoint functors

$$F : \text{Set}_* \rightleftarrows \text{Grp} : U$$

where U is forgetful functor and F is a *reduced* free group functor, that's it, for any pointed set X (with distinct point $*$), FX is a free group on X , quotient out by relation $* = 1$. The corresponding monad $UF : \text{Set}_* \rightarrow \text{Set}_*$ and its extension to the category of pointed simplicial sets sSet_* will be denoted by $F[-]$ and called *Milnor's construction*.

The similar monad, obtained from adjoint functors between Set_* and category of monoids Mon will be denoted by J - *James construction*, which form a subfunctor of $F[-]$.

Remark Traditionally, James construction is defined for topological space as a free reduced topological monoid on points of topological space. In the present paper James construction applied to simplicial set levelwise. Moreover, various functors and natural transformations between them will be considered for Set_* and sSet_* simultaneously, whenever it can not lead to the confusion.

Definition 2.1 The k -th (*classical*) *combinatorial James-Hopf map* is a natural transformation of functors

$$H_k : J \rightarrow J(-)^{\wedge k}$$

such that for any pointed set X , $H_k : JX \rightarrow JX^{\wedge k}$ defined as

$$H_k(x_1 \dots x_n) = \prod_{i_1 < \dots < i_k} x_{i_1} \wedge \dots \wedge x_{i_k}$$

where product is taken over all subsequences in $1..n$ (without repetitions of indices) arranged in the (right) lexicographic order. By functoriality the same map can be defined for simplicial sets levelwise.

The aim of this section is to extend the definition (2.1) in a natural way to the natural transformation $F[-] \rightarrow F(-)^{\wedge n}$ and give a motivation for such definition along with some basic properties. We will denote this extension by the same letter H_n whenever it cannot lead to a confusion.

Definition 2.2 The n -th combinatorial James-Hopf map is a natural transformation

$$H_n : F[-] \rightarrow F[(-)^{\wedge n}]$$

defined for any pointed (simplicial) set X on reduced words as

$$H_k(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}) = \prod_{(i_1 \dots i_k)} (x_{i_1} \wedge \dots \wedge x_{i_k})^{\varepsilon_{i_1} \dots \varepsilon_{i_k}}$$

here product is taken in (right) lexicographical order over sequences of indices $(i_1 \dots i_k)$ such that

$$i_j \leq i_{j+1} - \frac{\varepsilon_{i_{j+1}} + 1}{2}$$

that's it, product is taken over all subsequences $(i_1 \dots i_k)$ of $(1 \dots n)$ with possible repetition of indices and repetition of index i_j occurs only if corresponding exponent ε_{i_j} is negative.

Two immediate remarks here:

- (i) By construction (see commutator collection process below) H_k is well-defined for all (non necessary reduced) words in $F[X]$, for example:

$$* = H_k(*) = H_k(xx^{-1}) = (x \wedge \dots \wedge x)^{(-1)^{k-1}} (x \wedge \dots \wedge x)^{(-1)^k} = *$$

- (ii) H_k is a natural extension of a combinatorial James-Hopf map for James construction:

$$(4) \quad \begin{array}{ccc} F[X] & \xrightarrow{H_k} & F[X^{\wedge k}] \\ \uparrow & & \uparrow \\ J(X) & \xrightarrow{H_k} & J(X^{\wedge k}) \end{array}$$

To motivate this definition we follow the idea that Curtis apparently used in [4] to define a Hopf map H_2 in simplicial EHP sequence.

In Top_* there is an another map $h_k : \Omega \Sigma X \rightarrow \Omega \Sigma X^{\wedge k}$ called *Hilton-Hopf invariant* which is very close to a classical James-Hopf invariant H_k (in a sense that it induces the same map on homology and can be used in EHP-sequence), but defined only for co- H -spaces X , see [10]. We will use an extension of this definition to whole Top_* and afterwards translate it to sSet_* . To define h_k first recall the statement of the Hilton-Milnor's theorem as in [10]. For connected spaces X and Y loop-suspension over their wedge can be decomposed as a product

$$(5) \quad \Omega \Sigma(X \vee Y) \simeq \Omega \Sigma X \times \Omega \Sigma \left(\bigvee_{n \geq 0} Y \wedge X^{\wedge n} \right)$$

The second summand can be decomposed further and iterated application of this theorem gives the following decomposition

$$(6) \quad \Omega\Sigma(X \vee Y) \simeq \prod_{\omega \in \mathcal{B}(X, Y)} \Omega\Sigma(\omega(X, Y))$$

here $\mathcal{B}(X, Y)$ denotes the Hall basis of free Lie algebra on two letters X and Y and so each summand $\omega(X, Y)$ can be identified with a certain commutator bracket on two letters, written down as a smash product.

Let X be a co- H -space with comultiplication μ' . For the matter of convenience we will denote by A and B two identical copies of X . With this notation comultiplication will be a map $\mu' : X \rightarrow A \vee B$ and Hilton-Hopf map h_k is defined as a composition

$$(7) \quad \begin{array}{ccc} \Omega\Sigma X & \xrightarrow{h_k} & \Omega\Sigma X^{\wedge k} \\ \Omega\Sigma\mu' \downarrow & & \uparrow \pi_{B \wedge \dots \wedge A} \\ \Omega\Sigma(A \vee B) & \xrightarrow{\sim} & \prod_{\omega \in \mathcal{B}(A, B)} \Omega\Sigma(\omega(A, B)) \end{array}$$

here lower horizontal map is an inverse of the weak equivalence in Hilton-Milnor's theorem (6), right vertical map is a projection to summand, corresponded to Lie bracket $[B, A] \dots, A]$.

To extend h_k to whole Top_* we will introduce another map $\Omega\Sigma X \rightarrow \Omega\Sigma(X \vee X)$, as in [4], which is homotopy equivalent to $\Omega\Sigma\mu'$ in case of co- H -space X . Let i_A and i_B be two inclusions of X in $\Omega\Sigma(A \vee B)$, then $i_A * i_B$ is defined as a product of these inclusions in $\Omega\Sigma(A \vee B)$, i.e. extension of the composition

$$(8) \quad X \xrightarrow{\Delta} X \times X \xrightarrow{i_A \times i_B} \Omega\Sigma(A \vee B) \times \Omega\Sigma(A \vee B) \xrightarrow{\mu} \Omega\Sigma(A \vee B)$$

to an H -map $\Omega\Sigma X \rightarrow \Omega\Sigma A \vee B$

Lemma 2.3 For co- H -space X

$$i_A * i_B \simeq \Omega\Sigma\mu'$$

Proof Since extension of (8) to the H -map is unique up to homotopy, the statement follows from the commutativity of the following diagram, which can be checked explicitly:

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{i_A \times i_B} & \Omega\Sigma(A \vee B) \times \Omega\Sigma(A \vee B) & \xrightarrow{\mu} & \Omega\Sigma(A \vee B) \\ & \searrow \mu' & \uparrow & & \nearrow & & \\ & & X \vee X & & & & \end{array}$$

□

Now we can translate definition (7) to a simplicial group setting:

$$(9) \quad \begin{array}{ccc} F[X] & \xrightarrow{H_k} & F[X]^{\wedge k} \\ i_A * i_B \downarrow & & \uparrow \pi_{B \wedge A \wedge \dots \wedge A} \\ F[A] * F[B] & \xrightarrow{\cong} & \prod_{\omega \in \mathcal{B}(A, B)} F[\omega(A, B)] \end{array}$$

here $*$ denotes free product $F[A] * F[B] \cong F[A \vee B]$ and left vertical map sends generators x_i to $a_i b_i$ - product of generators of $F[A]$ and $F[B]$.

In Top_* Hilton-Milnor's decomposition given by a certain weak equivalence $\Omega \Sigma X \times \Omega \Sigma (\bigvee_{n \geq 0} Y \wedge X^{\wedge n}) \rightarrow \Omega \Sigma (X \vee Y)$ is constructed from iterated Whitehead products $[i_Y, i_X] \dots, i_X]$, so inverse to it (lower horizontal arrow) is somewhat mysterious. The situation is opposite in sSet_* : the map from free product $F[A] * F[B] \rightarrow \prod_{\omega \in \mathcal{B}(A, B)} F[\omega(A, B)]$ can be seen as a collection process in sense of Hall (see [7]), this allow us to describe the map purely combinatorially and it turns such map is a natural generalization of combinatorial James-Hopf map from free monoid to a free group. We briefly recall procedure of commutator collection applied to a free product $F[A] * F[B]$ from [4]. First for simplicity we assume that A (elements denoted by a_i) and B (elements denoted by b_j) are just pointed sets, after that the process can be extended to a simplicial sets by just applying it level-wise.

There is a natural projection $F[A] * F[B] \rightarrow F[A]$ (forgetting all b_j) with a section, induced by inclusion $A \rightarrow A \vee B$ and by Schreier's lemma the kernel is given by $F[B] * F[B \wedge F[A]]$, here free group on wedge $B \wedge F[A]$ is naturally identified as a subgroup of $F[A] * F[B]$ generated by commutators $[b, w]$ $b \in B, w \in F[A]$. The free summand $F[B \wedge F[A]]$ can be decomposed further using Tietze transformations (m is any number)

$$(10) \quad \begin{aligned} F[B \wedge F[A]] &\cong F[B \wedge A] * F[B \wedge A \wedge F[A]] \cong \dots \\ &\dots \cong F[B \wedge A] * F[B \wedge A \wedge A] * \dots * F[B \wedge A^{\wedge m} \wedge F[A]] \end{aligned}$$

this chain of isomorphisms can be seen as a process of expanding commutators of the form $[b, aw]$ as $[b, w][b, a][b, a, w]$, $b \in B, a \in A, w \in F[A]$. So, for any m after forgetting group structure there is a map

$$(11) \quad \begin{aligned} F[A] * F[B] &\rightarrow F[A] \times F[B] * F[B \wedge A] * \dots \\ &\dots * F[B \wedge A^{\wedge m}] * F[B \wedge A^{\wedge m} \wedge F[A]] \rightarrow F[A] \times F[\bigvee_{j=0}^m B \wedge A^{\wedge j}] \end{aligned}$$

given by (set-theoretic) retraction $F[A] * F[B] \rightarrow F[B] * F[B \wedge F[A]]$, composed with a chain of isomorphisms (10) and projection away from $F[B \wedge A^{\wedge m} \wedge F[A]]$. Hilton-Milnor's theorem for simplicial sets states that this map becomes a weak equivalence as $m \rightarrow \infty$ since connectivity of pieces $F[B \wedge A^{\wedge m} \wedge F[A]]$ grows as m increases. Same procedure can be applied to $F[B] * F[B \wedge F[A]]$ to split off $F[B]$ as direct summand and so on. Sequence of such procedures can be seen as Hall's commutator collection process which described in details in [7] and [4]. Here we only sketch the basic idea.

Let $w = a_1^{\varepsilon_1} b_1^{\eta_1} \dots a_n^{\varepsilon_n} b_n^{\eta_n}$ be a reduced word in $F[A] * F[B]$, $\varepsilon_i, \eta_j = \pm 1$. For simplicity first assume that all $\varepsilon_i = \eta_i = 1$. By pushing all a_i in the beginning of the word w , starting with a_1 , using identity

$$ca_i = a_i c[c, a_i], \quad c = b_j \text{ or } c = [b_j, a_{i_1} \dots a_{i_k}]$$

word w can be written as $w = w_A w'$, where $w_A = a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$ and w' consist of b_j and commutators of the form $[b_j, a_{i_1} \dots a_{i_k}]$, so $w' \in F[B \wedge F[A]]$ and mapping $w \rightarrow w'$ gives a retraction $F[A] * F[B] \rightarrow F[B] * F[B \wedge F[A]]$. In case of arbitrary exponents the following commutation rules should be applied:

$$(12) \quad c^{-1}a = a[c, a]^{-1}c^{-1}$$

$$(13) \quad ca^{-1} = a^{-1}c[c, a^{-1}] = a^{-1}c[c, a]^{-1}[c, a, a^{-1}]^{-1} = \dots$$

$$(14) \quad c^{-1}a^{-1} = a^{-1}[c, a^{-1}]^{-1}c^{-1} = a^{-1}[c, a, a^{-1}][c, a]c^{-1} = \dots$$

in the second and third rule here commutator identity $[x, y^{-1}] = [x, y]^{-1}[x, y, y^{-1}]^{-1}$ is applied to take negative exponent out of commutator (to move from $F[B \wedge F[A]]$ to $F[B \wedge A] * F[B \wedge A \wedge F[A]]$). Also note that exponents of commutators that lies inside $F[B \wedge A]$ are products of exponents of c and a .

Described process collect (arrange in (right) lexicographic order in the beginning of the word w) commutators of weight 0 (i.e. generators a_i), next iteration of Hilton-Milnor's theorem will collect all b_j , $[b_j, a_i]$ and so on. Now, using retraction mentioned above, we are ready to describe composition (9).

Remark Since in (9) the last map is a projection to summand $F[B \wedge A \wedge \dots \wedge A]$ (taking commutators of the form $[b_j, a_{i_1} \dots a_{i_k}]$), to define H_n only one retraction $w \rightarrow w'$ is needed:

$$(15) \quad \begin{array}{ccc} F[X] & \xrightarrow{H_n} & F[X^{\wedge n}] \\ \downarrow i_1 * i_2 & & \uparrow \pi_{B \wedge A \dots \wedge A} \\ F[A \vee B] & \xrightarrow{\rho} F[\bigvee_{j=0}^k B \wedge A^{\wedge j}] * F[B \wedge A^{\wedge k} \wedge F[A]] \longrightarrow \prod_{j=0}^k F[B \wedge A^{\wedge j}] \times F[B \wedge A^{\wedge k} \wedge F[A]] \end{array}$$

Difference between (9) and (15) is in order of commutators in each $F[B \wedge A^{\wedge k}]$. To ensure that new definition of H_k will contain old one for words with positive exponents, commutators needed to be arranged in (right) lexicographical order, which happens in further applications of Hilton-Milnor's decomposition. So in some sense (which will be made precise, see theorem (2.4) below) H_k defined uniquely only up to order of elements in the product in $F[B \wedge A^{\wedge k}]$

Fix $k > 0$ and consider $H_k(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n})$ as in (9). The image of $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in F[X]$ under $i_1 * i_2$ is $a_1^{\varepsilon_1} b_1^{\varepsilon_1} \dots a_n^{\varepsilon_n} b_n^{\varepsilon_n} \in F[A] * F[B]$. During collection process each $a_i^{\varepsilon_i}$ will give rise to commutators of the form $[b_j, a_{i_1} \dots a_{i_l}, a_i]^{\varepsilon_j \varepsilon_{i_1} \dots \varepsilon_{i_l} \varepsilon_i}$, where $j \leq i_1 \leq \dots \leq i_l \leq i$ and equality in this sequence appears only if corresponding exponent is -1 (rules (13) or (14) applied). After arranging such commutators for all i lexicographically and taking that of length k we get a definition (2.2).

Now we will show that product of all H_k can be seen as a non-abelian version of Magnus representation [9]. Let X be a pointed set and $\mathbb{Z}\langle\langle X \rangle\rangle$ be a ring of (non-commutative) formal power series on elements of X . *Magnus representation* μ is a homomorphism from free group on X to $\mathbb{Z}\langle\langle X \rangle\rangle$ defined on generators as

$$\mu : F[X] \rightarrow \mathbb{Z}\langle\langle X \rangle\rangle, \quad x \mapsto x + 1$$

By theorem of Magnus [9] this map is injective.

Ring $\mathbb{Z}\langle\langle X \rangle\rangle$ can be identified with an infinite product ($\bar{\mathbb{Z}}[-] = \mathbb{Z}[-]/\mathbb{Z}[*]$)

$$\prod_{n=0}^{\infty} \bar{\mathbb{Z}}[X]^{\otimes n} \cong \prod_{n=0}^{\infty} \bar{\mathbb{Z}}[X^{\wedge n}]$$

and for any k there are projections $\pi_k : \mathbb{Z}\langle\langle X \rangle\rangle \rightarrow \bar{\mathbb{Z}}[X^{\wedge k}]$ which maps formal power series to its elements of length k . By *length* of formal power series we mean the minimum among all lengths of its elements.

Theorem 2.4 *there is a commutative diagram*

$$(16) \quad \begin{array}{ccc} F[X] & \xhookrightarrow{\mu} & \mathbb{Z}\langle\langle X \rangle\rangle \\ H_k \downarrow & & \downarrow \pi_k \\ F[X^{\wedge k}] & \twoheadrightarrow_{\text{ab}} & \bar{\mathbb{Z}}[X^{\wedge k}] \end{array}$$

Proof Define $H_0 : F[X] \rightarrow \mathbb{Z}, x \mapsto 1$ and $H = \prod_{k=0}^{\infty} : F[X] \rightarrow \prod_{k=0}^{\infty} F[X^{\wedge k}]$. Then

the statement follows from commutativity of the following diagram

$$\begin{array}{ccc} F[X] & \xrightarrow{\mu} & \mathbb{Z}\langle\langle X \rangle\rangle \\ H \downarrow & \nearrow \text{ab} & \\ \prod_{k=0}^{\infty} F[X^{\wedge k}] & & \end{array}$$

For any $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in F[X]$ coefficients of power series $\mu(w) \in \mathbb{Z}\langle\langle X \rangle\rangle$ can be described as augmentations of Fox derivatives (see [6])

$$(17) \quad \mu(w) = \sum_{k=0}^{\infty} \sum_{(i_1 \dots i_k)} \varepsilon \partial_{i_1 \dots i_k}^k(w) x_{i_1} \dots x_{i_k}$$

here we use shorten notation $\partial_{i_1 \dots i_k}^k(w)$ for Fox derivative $\frac{\partial^k w}{\partial x_{i_1} \dots \partial x_{i_k}}$. Each of these given by formula ((3.5) in [6])

$$\partial_{i_1 \dots i_k}^k(w) = \sum_{(\lambda_1 \dots \lambda_k)} \varepsilon_{\lambda_1} \dots \varepsilon_{\lambda_k}$$

with summation over all sequences of indices $(\lambda_1 \dots \lambda_k)$ such that $i_{\lambda_j} = i_j$ for all $j = 1 \dots k$ and

$$\lambda_j \leq \lambda_{j+1} - \frac{\varepsilon_{\lambda_{j+1}} + 1}{2}$$

comparison it with definition (2.2) gives the result. \square

Corollary 2.5 (i) $H : F[X] \rightarrow \prod_{k=0}^{\infty} F[X^{\wedge k}]$ is injective

(ii) Let γ_n be a n -th term of lower central series of $F[X]$ and $\mathcal{L}^n(X) \cong \gamma_n / \gamma_{n+1}$ be a n -th Lie power, viewed as n -th homogeneous component of associated graded object, corresponded to lower central series filtration on $F[X]$. \bar{H}_n will denote composition of H_n with abelianization map $F[X]^{\wedge n} \rightarrow \bar{\mathbb{Z}}[X]^{\otimes n}$. Then the following diagram commute

$$\begin{array}{ccc} \gamma_n & \xrightarrow{\quad} & F[X] \\ \downarrow & & \downarrow \bar{H}_n \\ \mathcal{L}^n(X) & \xrightarrow{\quad} & \bar{\mathbb{Z}}[X]^{\otimes n} \end{array}$$

i.e. \bar{H}_n sends group commutators to ring commutators

Proof first statement is immediate, second one follows from the fact that elements of length n in $\mu(\gamma_n)$ forms $\mathcal{L}^n(X)$, see [9]. \square

3 Cohen group for the lower p -central series tower

In this section we define a Cohen group as a subgroup of natural transformations of functors $F[-] \rightarrow F^\wedge[-]$ which acts naturally on the lower p -central series tower, and therefore on the corresponding spectral sequence.

Let X be a (pointed) simplicial set and p be a prime. Recall that lower central series γ_n and lower p -central series $\gamma_n^{[p]}$ of $F[X]$ defined as a simplicial subgroups of $F[X]$ such that in each dimension k they are usual lower (p) -central series of a group $F[X]_k$:

$$(18) \quad (\gamma_n F[X])_k = \langle \{[g_1 \dots g_n] \mid g_i \in F[X]_k\} \rangle$$

$$(19) \quad (\gamma_n^{[p]} F[X])_k = \langle \{[g_1 \dots g_i]^{p^j} \mid g_i \in F[X]_k, ip^j \geq n\} \rangle$$

γ_n and $\gamma_n^{[p]}$ will be considered as subfunctors of $F[-]$.

Both series leads to a (functorial with respect to X) towers of fibrations of simplicial sets (denoted $\Gamma(X)$ and $\Gamma_p(X)$ respectively):

$$(20) \quad \begin{array}{ccc} F^\wedge[X] = \varprojlim F[X]/\gamma_n & & F_p^\wedge[X] = \varprojlim F[X]/\gamma_n^{[p]} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ F[X]/\gamma_{n+1} & \xleftarrow{\mathcal{L}^n(\bar{\mathbb{Z}}[X])} & F[X]/\gamma_{n+1}^{[p]} \xleftarrow{\mathcal{L}_{res}^n(\bar{\mathbb{Z}}/p[X])} \\ \downarrow & & \downarrow \\ F[X]/\gamma_n & & F[X]/\gamma_n^{[p]} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ F[X]/\gamma_2 = \bar{\mathbb{Z}}[X] & & F[X]\gamma_2^{[p]} = \bar{\mathbb{Z}}/p[X] \end{array}$$

Fibers of (20), denoted by $E^0\Gamma$ and $E^0\Gamma_p$ can be identified with Lie powers and restricted Lie powers on X and both towers leads to spectral sequences (convergence is due to Curtis [5])

$$(21) \quad E_{p,q}^1 = \pi_p \mathcal{L}^q(\bar{\mathbb{Z}}[X]) \Rightarrow \pi_{p+q} F[X], \quad E_{p,q}^1 = \pi_p \mathcal{L}^q(\bar{\mathbb{Z}}/p[X]) \Rightarrow \pi_{p+q} F[X] \otimes \mathbb{Z}/p$$

Remark Traditionally spectral sequence, constructed from lower p -central series filtration on Kan construction GX is called *unstable Adams spectral sequence* and was introduced in [1] (accelerated version) and [13]. Spectral sequence (21) is isomorphic to an unstable Adams spectral sequence for $G(\Sigma X)$ by a natural isomorphism of simplicial groups $F[X] \rightarrow G(\Sigma X)$.

(Levelwise) *Whitehead product* W_n for Milnor's construction is a homomorphism (a natural transformation, in fact) $W_n : F[X^{\wedge n}] \rightarrow F[X]$ defined on generators as

$$W_n(x_1 \wedge \dots \wedge x_n) = [x_1, x_2] \dots, x_n]$$

Now we can construct an analogue of Cohen group (3) for simplicial sets as a subgroup of $\text{Hom}_{\text{sSet}*}(F[-], F^{\wedge}[-])$ - (topological) group (under pointwise multiplication) of all simplicial natural transformations between Milnor's construction and its pro-nilpotent completion. Reason of choosing pro-nilpotent completion here is that original Cohen group H_{∞} was constructed as a pro-group and therefore contains certain infinite products, which can be crucial for realizing coalgebra idempotents.

Definition 3.1 *Cohen group* H_{∞} is a closure of a subgroup of $\text{Hom}_{\text{sSet}*}(F[-], F^{\wedge}[-])$ generated by compositions (evaluated at arbitrary X)

$$(22) \quad F[X] \xrightarrow{H_k} F[X^{\wedge k}] \xrightarrow{F[\bar{\Delta}]} F[X^{\wedge l}] \xrightarrow{F[\sigma]} F[X^{\wedge l}] \xrightarrow{W_l} F[X] \rightarrow F^{\wedge}[X]$$

for all positive $k \leq l$ and permutations $\sigma \in \Sigma_l$. Here H_k is James-Hopf map (2.2), $\bar{\Delta} : X^{\wedge k} \rightarrow X^{\wedge l}$ is iterated reduced diagonal, $\sigma : X^{\wedge l} \rightarrow X^{\wedge l}$ permuting factors in smash product and W_l is a Whitehead product.

To check that elements of H_{∞} induces maps between towers $\Gamma \rightarrow \Gamma$ and $\Gamma_p \rightarrow \Gamma_p$ it is sufficient to check that generators (22) preserves γ_n and $\gamma_n^{[p]}$ and since $F(\bar{\Delta})$, $F(\sigma)$ and W_l are homomorphisms only preservation of lower (p) -central series by H_k needed to be checked. Unfortunately, this is not a case as simplest example shows:

$$H_2([x, y]) = [(x \wedge x)^{-1}, (x \wedge y)^{-1}][(y \wedge y)^{-1}, (y \wedge x)](y \wedge x)^{-1}(x \wedge y)$$

To overcome this difficulty we define weighted lower central series filtration on $F[X^{\wedge m}]$ and show that James-Hopf map preserve lower (p) -central series in this weighted sense, which is sufficient to induce a maps of towers Γ and Γ_p .

Definition 3.2 *Weighted lower (p) -central series* $\gamma_n^w(F[X^{\wedge m}])$ and $\gamma_n^{[p],w}(F[X^{\wedge m}])$ are sequences of simplicial subgroups of $F[X]$ defined as follows:

- (i) if $n < m$ then $\gamma_n^w(F[X^{\wedge m}]) = F[X^{\wedge m}]$

(ii) if $n \geq m$. i.e. $n = qm + s$, $s < m$

$$\gamma_n^w(F[X^{\wedge m}]) = \gamma_{\lceil \frac{n}{m} \rceil}(F[X^{\wedge m}]) = \begin{cases} \gamma_q(F[X^{\wedge m}]), & \text{if } s = 0 \\ \gamma_{q+1}(F[X^{\wedge m}]), & \text{if } s > 0 \end{cases}$$

and similarly for $\gamma_n^{[p],w}(F[X^{\wedge m}])$

Lemma 3.3 *Whitehead products sends weighted lower (p -)central series to the usual one*

$$W_m(\gamma_n^w(F[X^{\wedge m}])) \subset \gamma_n, \quad W_m(\gamma_n^{[p],w}(F[X^{\wedge m}])) \subset \gamma_n^{[p]}$$

Proof Statement follows from the well known fact that $[\gamma_n, \gamma_m] \subset \gamma_{n+m}$. Mod- p version follows from the integral one:

$$W_m\left(\prod_{(i,j):ip^j \geq \lceil \frac{n}{m} \rceil} (\gamma_i)^{p^j}\right) = \prod_{(i,j):ip^j \geq \lceil \frac{n}{m} \rceil} W_m(\gamma_i)^{p^j} \subset \prod_{(i,j):ip^j \geq \lceil \frac{n}{m} \rceil} (\gamma_{im})^{p^j} \subset \prod_{(i,j):i'p^j \geq n} (\gamma_{i'})^{p^j}$$

where $i' = im$ in the last product □

Since we are working with free (simplicial) groups, it is convenient to switch our attention from lower (p -)central series γ_n and $\gamma_n^{[p]}$ to powers of (mod- p) augmentation ideals Δ^n and Δ_p^n . Recall that *augmentation ideal* and *(mod- p) augmentation ideal* are kernels of augmentation homomorphisms

$$\Delta \hookrightarrow \mathbb{Z}[F[X]] \xrightarrow{\epsilon} \mathbb{Z}, \quad \Delta_p \hookrightarrow \mathbb{Z}/p[F[X]] \xrightarrow{\epsilon} \mathbb{Z}/p$$

According to Magnus-Witt theorem γ_n and $\gamma_n^{[p]}$ can be expressed through Δ^n and Δ_p^n

$$\gamma_n = F[X] \cap (1 + \Delta^n), \quad \gamma_n^{[p]} = F[X] \cap (1 + \Delta_p^n)$$

We will use the following lemma (stated simultaneously for integral and mod- p case):

Lemma 3.4 *Let $f : A \rightarrow B$ be a pointed ($f(1) = 1$) map between free groups A and B such that $\mathbb{Z}f(\Delta^n) \subset \Delta^m$ ($\mathbb{Z}/pf(\Delta_p^n) \subset \Delta_p^m$). Then $f(\gamma_n) \subset \gamma_m$ (resp. $f(\gamma_n^{[p]}) \subset \gamma_m^{[p]}$) and there is a well defined map $\tilde{f} : A/\gamma_n \rightarrow B/\gamma_m$ (resp. $\tilde{f} : A/\gamma_n^{[p]} \rightarrow B/\gamma_m^{[p]}$)*

Proof We will prove only integral case. Using identification $\gamma_n = A \cap (1 + \Delta^n)$, $\forall g \in \gamma_n$ $g = 1 + a$, $a \in \Delta^n$. So $f(g) = \mathbb{Z}f(1 + a) = 1 + \mathbb{Z}f(a) \in (1 + \Delta^m) \cap B = \gamma_m$. For the second statement it is sufficient to show that $\forall g \in A$, $x \in \gamma_n$ $f(gx) \in f(g)\gamma_m$. Using the same identification $x = 1 + b$, $b \in \Delta^n \Rightarrow f(gx) = f(g) + \mathbb{Z}f(gb)$, so $f(gx) - f(g) \in \Delta^m$, $f(g)^{-1}f(gx) - 1 \in \Delta^m$, and finally $f(g)^{-1}f(gx) \in \gamma_m$ □

The standard machinery for working with powers of augmentation ideal is a Fox free differential calculus, first described in [6]. Here we will briefly sketch the definition of Fox derivatives and state the theorem of Fox, connecting these derivatives and powers of augmentation ideal.

Definition 3.5 Let FX be a free group on the set of generators X , $x_i \in X$ Fox derivative with respect to x_i is a linear map

$$\partial_i : \mathbb{Z}[FX] \rightarrow \mathbb{Z}[FX]$$

uniquely determined by the following properties

$$\begin{aligned} \partial_i(x_j) &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \\ \partial_i(ab) &= \partial(a)\varepsilon(b) + a\partial(b), \quad a, b \in \mathbb{Z}[FX] \end{aligned}$$

For a sequence of generators $x_{i_1} \dots x_{i_k}$, $\partial_{i_1 \dots i_k}^k$ will denote higher Fox derivative $\partial_{i_1} \circ \dots \circ \partial_{i_k}$ of order k . Fox derivative of order 0 is defined to be an augmentation homomorphism ε .

Theorem 3.6 (Fox, [6]) Let $a \in \mathbb{Z}[FX]$, then $a \in \Delta^n$ if and only if for all $0 \leq k < n$ and for all sequences of indices $(i_1 \dots i_k)$, $\varepsilon \partial_{i_1 \dots i_k}^k(a) = 0$, i.e. element of a group ring lies in the n -th power of augmentation ideal iff augmentations of all its Fox derivatives up to order $n - 1$ vanishes.

Remark This theorem can be seen as a corollary of formula (17) above, since image of Δ^n under Magnus embedding (extended linearly to a group ring) is $P_n \subset \mathbb{Z}\langle\langle X \rangle\rangle$ - formal power series of length $\geq n$.

Corollary 3.7 For $a \in \mathbb{Z}/p[FX]$, $a \in \Delta_p^n$ iff $\forall k, \forall (i_1 \dots i_k)$

$$\varepsilon \partial_{i_1 \dots i_k}^k(a) \equiv 0 \pmod{p}$$

Proof Corollary follows from the mod- p version of Magnus embedding

$$\mu_p : FX \rightarrow \mathbb{Z}/p\langle\langle X \rangle\rangle, \quad x \mapsto x + 1$$

which is the composition of μ and the quotient map $\mathbb{Z}\langle\langle X \rangle\rangle \rightarrow \mathbb{Z}/p\langle\langle X \rangle\rangle$. Therefore, coefficients of elements in the image of μ_p are given by augmentations of Fox derivatives modulo prime p . And as before, after extending μ_p to the group ring $\mathbb{Z}/p[FX]$, powers of augmentation ideals Δ_p^n will maps to a elements of length $\leq p$. These two facts give the result. \square

Theorem (3.6) together with lemma (3.4) provides a convenient way to check whenever H_k preserves weighted lower (p -)central series filtration.

Theorem 3.8 *Simplicial James-Hopf map $H_m : F[X] \rightarrow F[X^{\wedge m}]$ sends lower central series to a weighted one:*

$$H_m(\gamma_n) \subset \gamma_n^w, H_m(\gamma_n^{[p]}) \subset \gamma_n^{[p],w}$$

Proof By lemma (3.4) it is sufficient to prove preservation of weighted augmentation ideal filtration, i.e. that $\forall g_1, \dots, g_m \in F[X] \ H_n((g_1 - 1) \dots (g_m - 1)) \in \Delta_w^m$. After writing each g_i in normal form, the statement can be reduced to the case when all g_i are generators of $F[X]$, possibly with negative exponent. Since H_n is natural with respect to maps between sets of generators $Y \rightarrow X$, all this generators can be taken distinct from each other and ordered lexicographically.

$$\begin{array}{ccc} \mathbb{Z}F[Y] & \xrightarrow{\mathbb{Z}H_n} & \mathbb{Z}F[Y^{\wedge n}] \\ f \downarrow & & \downarrow f \wedge \dots \wedge f \\ \mathbb{Z}F[X] & \xrightarrow{\mathbb{Z}H_n} & \mathbb{Z}F[X^{\wedge n}] \end{array}$$

So

$$\mathbb{Z}H_n((x_{i_1}^{\varepsilon_{i_1}} - 1) \dots (x_{i_m}^{\varepsilon_{i_m}} - 1)) = (f \wedge \dots \wedge f) \mathbb{Z}H_n((y_1^{\varepsilon_1} - 1) \dots (y_m^{\varepsilon_m} - 1))$$

According to the theorem (3.6) it is sufficient to show that augmentations of all derivatives $\partial_{J_i, \dots, J_1}^i$ of $\mathbb{Z}H_n((y_1^{\varepsilon_1} - 1) \dots (y_m^{\varepsilon_m} - 1))$ of orders up to q (or $q - 1$) are vanished for all possible combinations of multi-indices $J_k = y_{k_1} \wedge \dots \wedge y_{k_n}$.

Let $Y = y_1^{\varepsilon_1} \dots y_m^{\varepsilon_m}$, $|v|$ is the length of word v and notations $s \subset v$, $s \subseteq v$ will stands for s being a subword of word v and s being a subword (with possible repetitions) of word v respectively. With these notations

$$(23) \quad \varepsilon \partial_{J_i, \dots, J_1}^i \mathbb{Z}H_n((y_1^{\varepsilon_1} - 1) \dots (y_m^{\varepsilon_m} - 1)) = \sum_{j=0}^m (-1)^{m-j} \sum_{v \subset Y, |v|=j} \varepsilon(\partial_{J_i, \dots, J_1}^i H_n(v))$$

According to formula (3.5) of [6], in which higher derivatives of arbitrary reduced word in free group are computed, and using formula for $H_n(v) = \prod_{d \subseteq v} d^{\varepsilon_d}$ we see that

$$(24) \quad \varepsilon \partial_{J_i, \dots, J_1}^i (H_n(v)) = \sum \varepsilon_{d_1} \dots \varepsilon_{d_i}$$

with sum over all sequences of subwords (with possible repetitions) d_k of v , such that for all subwords d_k in sequence $J_k = d_k$ and subword d_k can repeat if the corresponding exponent ε_{d_k} is negative.

This means that augmentation of derivative is not zero if corresponding v contains all subwords J_k (again, repetitions are allowed if corresponding exponents are negative). First of all, since everything in v is ordered lexicographically, all derivatives with respect to tuples $J_q \dots J_1$, which are not in (anti)lexicographic order, are zero. Second, since all elements in Y are different, all subwords $d \subseteq v$ are different from each other, so all J_k appears in $H_n(v)$ only once, with fixed exponent ε_{J_k} and (23) will take form

$$\varepsilon \partial_{J_i, \dots, J_1}^i \mathbb{Z}H_n((y_1^{\varepsilon_1} - 1) \dots (y_m^{\varepsilon_m} - 1)) = \varepsilon_{J_i} \dots \varepsilon_{J_1} \sum_{j=0}^m (-1)^{m-j} \sum_{v \subset Y, |v|=j, J_i \dots J_1 \subseteq v} 1$$

Next we count the number of subwords $v \subset Y$, of length j which contains $J_i \dots J_1 \subseteq v$, for fixed tuple $J_i \dots J_1$. If generator y_l have a positive exponent in Y then derivative with respect to a tuple, which contains multi-index $J_p = y_{p_1} \wedge \dots \wedge y_l \wedge y_l \wedge \dots \wedge y_{p_n}$ with repetition of y_l will be zero. So we can assume that all J_k have repetition of indexes inside only if the exponent of corresponding generator is negative in Y . Same goes for the case when $J_k = J_{k+1}$ - such derivatives a zero if the product of exponents of generators in Y which also form J_k is positive (does not depend on v). With this limitations, the number of v of length j such that $J_i \dots J_1 \subseteq v$ is precisely

$$\binom{m - n_1 - \dots - n_r}{j - n_1 - \dots - n_r}$$

Here r is a total number of different J_k ($1 \leq r \leq i$) and n_k is a total number of different generators in J_k ($1 \leq n_k \leq n$). Denoting $N = n_1 + \dots + n_r$ and substituting this number in the formula we have

$$\begin{aligned} \varepsilon \partial_{J_i, \dots, J_1}^i \mathbb{Z}H_n((y_1^{\varepsilon_1} - 1) \dots (y_m^{\varepsilon_m} - 1)) &= \varepsilon_{J_i} \dots \varepsilon_{J_1} \sum_{j=N}^m (-1)^{m-j} \binom{m-N}{j-N} = \\ &= \begin{cases} 0 & m - N \neq 0 \\ \varepsilon_{J_i} \dots \varepsilon_{J_1} & m - N = 0 \end{cases} \end{aligned}$$

since $N \leq in$, $s < n$ and $i \leq q$, last case can only happen if $s = 0, N = qn$ which is exactly case 1 in the definition of weighted filtration. \square

Remark Passi (see [11], [12]) defined a map f between group G and R -module M to be *polynomial of degree $\leq n$* if its extension on a group algebra $R[G]$ vanishes on Δ_R^{n+1} . Therefore theorem (3.8) states that maps $\varepsilon \partial_{J_j \dots J_1}^j \mathbb{Z}H_k : \mathbb{Z}F[X] \rightarrow \mathbb{Z}$ are \mathbb{Z} -polynomial of degree $\leq jk$.

Corollary 3.9 *James-Hopf invariants H_k sends lower p -central series to a weighted ones:*

$$H_k(\gamma_n^{[p]}) \subset \gamma_n^{[p],w}$$

Proof As before we switch our attention to the powers of augmentation ideal Δ_p , by corollary (3.7) we need to show that augmentations of corresponding Fox derivatives of H_k vanishes mod p . This follows from proposition (1.11) in [12] which states that composition of \mathbb{Z} -polynomial map $f : G \rightarrow A$ and homomorphism $\theta : A \rightarrow M$ for some R -module M is a R -polynomial map. Application of this proposition to $\varepsilon \partial_{j_1, \dots, j_1}^j H_k$ and quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/p$ gives the result. \square

Theorem 3.10 *Elements of Cohen group $H_\infty(X)$ induce maps of towers $\Gamma \rightarrow \Gamma$ and $\Gamma_p \rightarrow \Gamma_p$, and hence equip spectral sequences (21) with an action of $H_\infty : E_{p,q}^r \rightarrow E_{p,q}^r$ which preserves differential.*

Proof Combining lemma (3.3) with theorem (3.8) we see that generators of Cohen group (and hence all elements) preserves lower (p -)central series filtration, and it is immediate that inclusion $F[-] \rightarrow F^\wedge[-]$ induce an isomorphisms

$$(25) \quad F[-]/\gamma_n \cong F^\wedge[-]/\gamma_n, \quad F[-]/\gamma_n^{[p]} \cong F^\wedge[-]/\gamma_n^{[p]},$$

so there are a well defined maps

$$(26) \quad H_\infty \rightarrow \text{Hom}(\Gamma, \Gamma), \quad f : F[-] \rightarrow F^\wedge[-] \mapsto \{f_n\}_n : \{F[-]/\gamma_n\}_n \rightarrow \{F[-]/\gamma_n\}_n$$

$$(27) \quad H_\infty \rightarrow \text{Hom}(\Gamma_p, \Gamma_p), \quad f : F[-] \rightarrow F^\wedge[-] \mapsto \{f_n\}_n : \{F[-]/\gamma_n^{[p]}\}_n \rightarrow \{F[-]/\gamma_n^{[p]}\}_n$$

Note here that $\text{Hom}(\Gamma, \Gamma)$ and $\text{Hom}(\Gamma_p, \Gamma_p)$ are groups with respect to pointwise multiplication on each layer of tower, and therefore (26) are homomorphisms. here f_n denotes induced map on quotients $F[-]/\gamma_n$ (or $F[-]/\gamma_n^{[p]}$). Image of these maps will be called *Cohen groups for towers Γ and Γ_p* , denoted by $H\Gamma_\infty$ and $H\Gamma_{p,\infty}$

Action of H_∞ on spectral sequences comes from the corresponding action on towers through an induced action on exact couples in a standard way. Each $f \in H_\infty$ gives a map $E_{p,q}^1 \rightarrow E_{p,q}^1$ which is an induced map on homotopy groups of fibers. Preservation of differential follows from the fact that f induce a map of towers. \square

4 Functorial decomposition of lower p -central series spectral sequence

Through this section $\mathbf{k} = \mathbb{Z}/p$ will be ground field with p elements, V be a \mathbf{k} -module. First we will review few facts about natural coalgebra transformations $T \rightarrow T$. Tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ can be considered as a (connected) Hopf algebra (see [8]) with the standard multiplication m and comultiplication ψ defined as

$$\psi(v) = v \otimes 1 + 1 \otimes v, \quad v \in V$$

on generators (making them primitive) and extended to all $T(V)$. Conjugation s defined as

$$s(x_1 \dots x_k) = (-1)^k x_k \dots x_1$$

Using this comultiplication and conjugation one can define a group structure on set of all coalgebra maps $\text{Hom}_{\text{Coalg}_{\mathbf{k}}}(T(V), T(V))$, using so-called *convolution product*: for $f, g : T(V) \rightarrow T(V)$ $f * g$ defined as composition

$$(28) \quad T(V) \xrightarrow{\psi} T(V) \otimes T(V) \xrightarrow{f \otimes g} T(V) \otimes T(V) \xrightarrow{m} T(V)$$

Note that this product is a complete analogue of point-wise multiplication on set of all maps $G \rightarrow G$ for group G :

$$G \xrightarrow{\Delta} G \times G \xrightarrow{f \times g} G \times G \xrightarrow{\mu} G$$

T as coalgebra naturally filtered by *James filtration*:

$$J_k = \bigoplus_{n=0}^k T_n$$

here T_n stands for a functor $T_n(V) = V^{\otimes n}$ and this filtration is complete and gives $\text{Hom}_{\text{Coalg}_{\mathbf{k}}}(T(V), T(V))$ structure of a pro-group.

Lemma 4.1 ((2.4) in [14]) *there is a short exact sequence of groups*

$$(29) \quad \text{Hom}_{\mathbf{k}}(T_n, \mathcal{L}^n) \cong \ker \hookrightarrow \text{Hom}_{\text{Coalg}_{\mathbf{k}}}(J_n, T) \twoheadrightarrow \text{Hom}_{\text{Coalg}_{\mathbf{k}}}(J_{n-1}, T)$$

For a simplicial realization theorem below we will use a specific set of topological generators of a pro-group $\text{Hom}_{\text{Coalg}_{\mathbf{k}}}(T, T)$, first apparently appeared in [3]:

Theorem 4.2 Any natural coalgebra self-map $f_V : T(V) \rightarrow T(V)$ (evaluated at V) can be written down as a (possibly infinite) convolution product $\prod_{k=1}^{\infty} f_k$ of the maps of the form

$$(30) \quad T(V) \xrightarrow{H_k^{alg}} T(V^{\otimes k}) \xrightarrow{T(\sigma)} T(V^{\otimes k}) \xrightarrow{\beta_k^T} T(V)$$

here H_k^{alg} is an algebraic James-Hopf map (proposition (5.3) in [3]), $\sigma \in \Sigma_k$ and β_k^T is a Hopf extension of a map $\beta_k : V^{\otimes k} \rightarrow T(V)$, $\beta_k(x_1 \dots x_k) = [x_1 \dots x_k]$.

Proof It is sufficient to show the statement for each element in inverse system. The statement is trivial for maps in $\text{Hom}_{\text{Coalg}_k}(J_1, T)$ and suppose that any element $f \in \text{Hom}(J_{k-1}, T)$ can be expressed as

$$f = \prod_{j=1}^{k-1} f_j$$

Given $f : J_k \rightarrow T$ it can be decomposed in a convolution product $f_1 * f_2$, where $f_1 : J_k \rightarrow T$ is a coalgebra map which is an identity map on the $(J_k)_0 = \mathbf{k}$, trivial on $(J_k)_i$, $0 < i < k$ and $(f_1)_k = (f)_k$ and f_2 is the restriction of f to J_{k-1} . Therefore it is sufficient to express f_1 as a convolution product of the maps of the form (30). We will use universal property of the algebraic James-Hopf map which is direct analogue of the universal property of the classical combinatorial James-Hopf map:

$$(31) \quad \text{Top}_* : \quad \text{Coalg} :$$

$$\begin{array}{ccc} J_k X & \hookrightarrow & JX \\ \downarrow & & \downarrow H_k \\ X^{\wedge k} & \hookrightarrow & JX^{\wedge k} \end{array} \quad \begin{array}{ccc} J_k V & \hookrightarrow & T(V) \\ \downarrow & & \downarrow H_k^{alg} \\ V^{\otimes k} & \hookrightarrow & T(V^{\otimes k}) \end{array}$$

Applying it to f_1 (which factors through T_k by construction) we see that

$$\begin{array}{ccc} J_k & \hookrightarrow & T \\ \downarrow & \searrow f_1 & \downarrow H_k^{alg} \\ T_k & \xrightarrow{\bar{f}_1} & T(T_k) \end{array}$$

T

$T(\bar{f}_1)$

Since $\text{Hom}_{\mathbf{k}}(T_k, T_l) = \mathbf{k}[\Sigma_k]$ if $k = l$ and 0 otherwise, the map $\bar{f}_1 : T_k \rightarrow T$ is completely determined by a certain element $\sum_i a_i \sigma_i$ of a group algebra of the

symmetric group Σ_k . Moreover, since \bar{f}_1 came initially from a coalgebra map, its image lies in \mathcal{L}_{res}^k (see proposition (2.4) in [14]) and therefore it factors through β_k . Finally under isomorphism in (29) sum of permutations maps to a product and we have

$$f_1 = \prod_i \beta_k^T \circ \sigma_i^{a_1} \circ H_k$$

□

Remark It can be noticed that in contrast with definition (22) there are no reduced diagonals in generators of $\text{Hom}_{\text{Coalg}_{\mathbf{k}}}(T, T)$ - these maps are trivial on tensor algebra.

Similarly to l.c.s. tower, using powers of augmentation ideal filtration on a group algebra $\mathbf{k}[F[-]]$ one can form a tower of simplicial abelian groups (denoted by $\mathbf{k}\Gamma_p$)

$$(32) \quad \begin{array}{c} \mathbf{k}[F]^\wedge = \varprojlim \mathbf{k}[F]/\Delta_p^n \\ \downarrow \\ \vdots \\ \downarrow \\ \mathbf{k}[F]/\Delta_p^{n+1} \longleftarrow T_n(\bar{\mathbf{k}}[-]) \\ \downarrow \\ \mathbf{k}[F]/\Delta_p^n \\ \downarrow \\ \vdots \\ \downarrow \\ \mathbf{k}[F]/\Delta_p^2 = \mathbf{k} \oplus \bar{\mathbf{k}}[-] \end{array}$$

with fibers identified with a homogeneous components of tensor algebra functor $T(\bar{\mathbf{k}}[-]) : \mathbf{sSet}_* \rightarrow \mathbf{sMod}_{\mathbf{k}}$ (analogue of Magnus-Witt theorem, see [4])

$$T_n(\bar{\mathbf{k}}[-]) \xrightarrow{\cong} \Delta_p^n / \Delta_p^{n+1}, \quad x_1 \dots x_n \mapsto (x_1 - 1) \dots (x_n - 1) + \Delta_p^{n+1}$$

The natural inclusion $F[-] \rightarrow \mathbf{k}[F[-]]$ of Milnor's construction to its group algebra can be extended to a level of towers

$$F[-]/\gamma_n^{[p]} \rightarrow \mathbf{k}[F[-]]/\Delta_p^n$$

in a way that on the level of fibers one get inclusion $\mathcal{L}_{res}^n \rightarrow T_n$ of homogeneous components of free restricted Lie algebra to homogeneous components of tensor algebra as primitive elements in degree n (see also [8]).

As before E^0 will denote the functor of fibers of tower, in case of towers (20) and (32) it takes values in graded objects of $\mathbf{sMod}_{\mathbf{k}}$. Therefore there is a map

$$(33) \quad \mathrm{Hom}_{\mathbf{sSet}_*}(\Gamma_p, \Gamma_p) \xrightarrow{E^0 \circ \mathbf{k}[-]} \mathrm{Hom}_{\mathbf{sCoalg}_{\mathbf{k}}}(T(\bar{\mathbf{k}}[-]), T(\bar{\mathbf{k}}[-]))$$

Theorem 4.3 (33) is a well-defined epimorphism of groups, i.e. every coalgebra natural transformation of functor $T(\bar{\mathbf{k}}[-])$ to itself can be lifted up to a natural self-transformation of tower $\Gamma_p \rightarrow \Gamma_p$

Proof First we will show that $E^0 \circ \mathbf{k}$ is a well defined homomorphism, i.e. for any $f : \Gamma_p \rightarrow \Gamma_p$ $E^0(\mathbf{k}[f])$ is a coalgebra map and products are preserved. To see this we pass to the limits of the towers:

$$\begin{array}{ccccc} \mathrm{Hom}(\Gamma_p, \Gamma_p) & \xrightarrow{\mathbf{k}} & \mathrm{Hom}(\mathbf{k}\Gamma_p, \mathbf{k}\Gamma_p) & \xrightarrow{E^0} & \mathrm{Hom}(T, T) \\ \downarrow \varprojlim & & \downarrow \varprojlim & \nearrow E^0 & \\ \mathrm{Hom}(F_p^\wedge[-], F_p^\wedge[-]) & \xrightarrow{\varprojlim \mathbf{k}} & \mathrm{Hom}(\mathbf{k}[F]^\wedge, \mathbf{k}[F]^\wedge) & & \end{array}$$

here we consider $\mathbf{k}[F]^\wedge$ as a complete coalgebra with a filtration, induced from Δ_p -adic filtration on $\mathbf{k}[F]$ and E^0 functor in this context is a functor of passing to associate graded coalgebra. E^0 respects (completed) tensor products:

$$(34) \quad E^0(-\hat{\otimes}-) = E^0(- \otimes -) = E^0(-) \otimes E^0(-)$$

$\varprojlim \mathbf{k}$ is an extension of group algebra functor $\mathbf{k}[-] : \mathrm{Grp} \rightarrow \mathrm{Hopf}_{\mathbf{k}}$ to completions and therefore for any map $g : F_p^\wedge \rightarrow F_p^\wedge$ $\varprojlim \mathbf{k}(g)$ is a map of complete coalgebras. This shows that the map (33) is well-defined.

By (34) and since \varprojlim commutes with direct products and $\varprojlim \mathbf{k}(F_p^\wedge \times F_p^\wedge) = \mathbf{k}F^\wedge \hat{\otimes} \mathbf{k}F^\wedge$ $E^0 \circ \mathbf{k}$ sends pointwise multiplication

$$F/\gamma_n^{[p]} \xrightarrow{\Delta} F/\gamma_n^{[p]} \times F/\gamma_n^{[p]} \xrightarrow{f \times g} F/\gamma_n^{[p]} \times F/\gamma_n^{[p]} \xrightarrow{\mu} F/\gamma_n^{[p]}$$

to the convolution product (28).

To prove that $E^0 \circ \mathbf{k}[-]$ is surjective we will show that generators of Cohen group for towers maps to generators of $\mathrm{Hom}_{\mathbf{sCoalg}_{\mathbf{k}}}(T(\bar{\mathbf{k}}[-]), T(\bar{\mathbf{k}}[-]))$, which are described

in (4.2). It is clear that $E^0(\mathbf{k}[F[\sigma]]) = T(\sigma)$ and $E^0(\mathbf{k}[W_n]) = \beta_n^T$. Only universal property (31) of H_k^{alg} is used in theorem (4.2) and $E^0(\mathbf{k}[H_k])$ satisfy it since $E^0(\mathbf{k}[J]) \equiv E^0(\mathbf{k}[F])$ - associated graded of free group ring and free monoid ring are naturally isomorphic (isomorphism induced by inclusion $J \rightarrow F$). The result follows from diagrams (31) and (4). \square

We proceed to a decomposition of spectral sequence (21). First we translate the well-known result about homotopy idempotents of H -spaces to \mathbf{sSet}_* .

Lemma 4.4 *Let G be a connected simplicial group and $f : G \rightarrow G$ be a simplicial self-map, such that*

$$f_* : \pi_* G \rightarrow \pi_* G$$

is an idempotent. Then, as simplicial set, G is weak equivalent to a product $A \times B$,

$$A = \operatorname{colim} f, \quad B = \operatorname{colim} g$$

where $g = [\operatorname{id}]f^{-1}$ denotes a complement to f in the group $\operatorname{Hom}_{\mathbf{sSet}_}(G, G)$*

Proof f_* is a (graded) idempotent element of $\operatorname{End}(\bigoplus_i \pi_i G)$ and therefore there is an isomorphism

$$(35) \quad \bigoplus_i \pi_i G \equiv \bigoplus_i \operatorname{im} f_i \oplus \operatorname{im}(\operatorname{id} - f_i)$$

here $\operatorname{im} f_i = \operatorname{colim} f_i = \pi_i \operatorname{colim} f$, $\operatorname{im}(\operatorname{id} - f_i) = \operatorname{colim}(\operatorname{id} - f_i) = \pi_i \operatorname{colim}[\operatorname{id}][f^{-1}]$ and isomorphism (35) can be realized by

$$G \xrightarrow{\Delta} G \times G \rightarrow A \times B$$

\square

Well known results about filtered colimits and fibrations in \mathbf{sSet}_* will be organized in the next lemma for simplicity.

Lemma 4.5 *Let $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X$ be a sequence of simplicial sets and simplicial maps between them with colimit X and*

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X \\ \downarrow f_0 & & \downarrow f_1 & & & & \downarrow f \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \dots & \longrightarrow & Y \end{array}$$

be a map of such sequences. Then:

- (i) $\pi_* X = \operatorname{colim} \{ \pi_*(X_0) \rightarrow \pi_*(X_1) \rightarrow \dots \}$
- (ii) if f_n are Kan fibrations for all n then f is also Kan fibration.
- (iii) the fiber of f is a colimit of fibers of f_n

Given arbitrary natural coalgebra decomposition of tensor algebra $T \simeq A \otimes B$, it can be extended to \mathbf{sSet}_* by switching from functor $T : \mathbf{Mod}_{\mathbf{k}} \rightarrow \mathbf{Coalg}_{\mathbf{k}}$ to $T(\bar{\mathbf{k}}[-]) : \mathbf{sSet}_* \rightarrow \mathbf{sCoalg}_{\mathbf{k}}$. For any $X \in \mathbf{sSet}_*$ composition of retraction and inclusion

$$f : T(\bar{\mathbf{k}}[X]) \xrightarrow{r_A} A(\bar{\mathbf{k}}[X]) \xrightarrow{i_A} T(\bar{\mathbf{k}}[X])$$

is an idempotent in $\operatorname{Hom}_{\mathbf{sCoalg}_{\mathbf{k}}}(T(\bar{\mathbf{k}}[X]), T(\bar{\mathbf{k}}[X]))$ and $A(\bar{\mathbf{k}}[X]) \simeq \operatorname{colim} f$. This idempotent give rise to a map of towers $\varphi : \Gamma_p \rightarrow \Gamma_p$ by simplicial realization theorem (4.3), on each level n denoted by $\varphi_n : F[X]/\gamma_n^{[p]} \rightarrow F[X]/\gamma_n^{[p]}$.

Theorem 4.6 For any natural (simplicial) coalgebra decomposition $T \simeq A \otimes B$ there exists towers of fibrations \mathcal{A} and \mathcal{B} such that

$$\Gamma_p \simeq \mathcal{A} \times \mathcal{B}$$

and such that its fibers are functorially given by primitive elements of simplicial coalgebras A and B :

$$E^0(\mathcal{A}) \simeq PA, \quad E^0(\mathcal{B}) \simeq PB$$

Proof Given an $X \in \mathbf{sSet}_*$, and decomposition $T(\mathbf{k}[X]) \simeq A(\mathbf{k}[X]) \otimes B(\mathbf{k}[X])$ let $f_X : T(\mathbf{k}[X]) \rightarrow T(\mathbf{k}[X])$ be an idempotent, corresponding to $A(\mathbf{k}[X])$, as before, its realization on the level of towers is $\varphi_X : \Gamma_p(X) \rightarrow \Gamma_p(X)$. Since in each dimension k X_k is a finite set, for any n $(F[X]/\gamma_n^{[p]})_k$ are finite groups. Hence, after sufficiently many iterations $\varphi_{X,n} : F[X]/\gamma_n^{[p]} \rightarrow F[X]/\gamma_n^{[p]}$ will become an idempotent, i.e. there exist number $N(n, k)$, such that

$$(36) \quad (\varphi_{X,n})_k^{N(n,k)} : (F[X]/\gamma_n^{[p]})_k \rightarrow (F[X]/\gamma_n^{[p]})_k$$

are idempotents. Since φ_X was initially map of towers, φ_X^N is also one, which follows from commutativity of the following diagrams:

$$\begin{array}{ccccc}
 (F[X]/\gamma_n^{[p]})_k & \begin{array}{c} \xrightarrow{\dots} \\ \xleftarrow{\dots} \end{array} & (F[X]/\gamma_n^{[p]})_{k-1} & & F[X]/\gamma_n^{[p]} \xrightarrow{(\varphi_{X,n})_k^{N(n,k)}} F[X]/\gamma_n^{[p]} \\
 \downarrow (\varphi_{X,n})_k^{N(n,k)} & & \downarrow (\varphi_{X,n})_k^{N(n,k-1)} & & \downarrow \\
 (F[X]/\gamma_n^{[p]})_k & \begin{array}{c} \xrightarrow{\dots} \\ \xleftarrow{\dots} \end{array} & (F[X]/\gamma_n^{[p]})_{k-1} & & F[X]/\gamma_{n-1}^{[p]} \xrightarrow{(\varphi_{X,n-1})_k^{N(n-1)}} F[X]/\gamma_{n-1}^{[p]}
 \end{array}$$

Now lemma (4.4) is applied to groups $F[X]/\gamma_n^{[p]}$ and idempotents (36) to get a decomposition

$$(37) \quad F[X]/\gamma_n^{[p]} \simeq \mathcal{A}_n(X) \times \mathcal{B}_n(X), \quad \mathcal{A}_n(X) = \operatorname{colim} \varphi_{X,n}^{N(n)} = \operatorname{colim} \varphi_{X,n}$$

This weak equivalence is consistent with a maps $F[X]/\gamma_n^{[p]} \longrightarrow F[X]/\gamma_{n-1}^{[p]}$ and by lemma (4.5) the natural maps $\mathcal{A}_n(X) \rightarrow \mathcal{A}_{n-1}(X)$ are fibrations, similarly for $\mathcal{B}_n(X)$, so (37) gives indeed decomposition of the whole tower $\Gamma_p(X)$, and this decomposition is functorial with respect to X .

By commutativity of the following diagram

$$\begin{array}{ccc} \operatorname{Hom}(\Gamma_p, \Gamma_p) & \xrightarrow{\mathbf{k}} & \operatorname{Hom}(\mathbf{k}\Gamma_p, \mathbf{k}\Gamma_p) \\ E^0 \downarrow & & \downarrow E^0 \\ \operatorname{Hom}(\mathcal{L}, \mathcal{L}) & \xleftarrow{P} & \operatorname{Hom}(T, T) \end{array}$$

and lemma (4.5):

$$E^0 \mathcal{A} = E^0 \operatorname{colim} \varphi \simeq \operatorname{colim} E^0 \varphi = \operatorname{colim} Pf = P(\operatorname{colim} f) = PA$$

□

Remark After passing to the limits of towers one have functorial decomposition of the pro- p completion of Milnor's construction $F[-]$

$$(38) \quad F_p^\wedge \simeq \varprojlim \mathcal{A} \times \varprojlim \mathcal{B}$$

and since $|F_p^\wedge[X]| \simeq_p \Omega\Sigma|X|$ (38) can be seen as a translation of the classical decomposition (1) to the simplicial setting.

Corollary 4.7 *Any natural coalgebra decomposition $T \simeq A \otimes B$ induce a decomposition of spectral sequence*

$$(39) \quad E_{p,q}^1 = \pi_p \mathcal{L}_{res}^q \Rightarrow \pi_{p+q} F, \quad E_{p,q}^r = E_{p,q}^r(A) \oplus E_{p,q}^r(B)$$

as a functor on \mathbf{sSet}_* , with first pages of $E_{p,q}^r(A)$, $E_{p,q}^r(B)$ given by homotopy groups of primitive elements of simplicial coalgebras A and B :

$$E_{p,q}^1(A) = \pi_p(PA)_q, \quad E_{p,q}^1(B) = \pi_p(PB)_q$$

In conclusion we will formulate the result about accelerated functorial sub-spectral sequence of the unstable Adams spectral sequence for suspensions, which is the direct corollary of (4.7)

Corollary 4.8 *Let A^{\min} be a minimal functorial coalgebra retract of T and $T = A^{\min} \otimes B^{\max}$ be a corresponding decomposition. Then sub-spectral sequence $E_{p,q}^r(A^{\min})$ of $E_{p,q}^r$ have non-trivial cells only in columns with numbers p^k*

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